THE FLOW ABOUT AN AERODYNAMIC CASCADE OF THIN OSCILLATING SECTIONS

(OBTEKANIE AERODINAMICHESKOI RESHETKI TONKIKH VIBRIRUIUSHCHIKH PROFILEI)

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Various cases of the oscillation of cascades with constant circulation have been discussed by Sedov [1] , and synchronous in-phase oscillations of the sections in the cascade have been investigated in the works of Khaskind [2]; Sirazetdinov [3]; Songen [4,5]; Chang and Chu [6]; Nickel [7]; Woods [8] and Popescu [9,10]. In some of these works the problem has also been examined more broadly. Legendre [11] and Timman [12] especially have investigated the case of the out-of-phase oscillation of adjacent blades. Sisto [13] studied synchronous oscillations with constant phase shift. Meister [14,15] solves the problem of independent oscillation of the sections in the cascade.

In this paper unsteady flows of an incompressible fluid through a cascade whose adjacent sections are oscillating with different frequencies, phases and amplitudes are investigated by means of the acceleration potential. Translational and rotational oscillations are examined.

Added masses and forces which have a circulational nature are found.

Some well-known exact solutions are obtained as special cases.

In a number of cases approximations, which are compared with the exact expressions, are introduced.

As the pitch of the cascade increases without limit the formulas obtained reduce to the well-known solutions for an isolated oscillating wing [1,16].

1. Statement of the problem. We shall examine the planar flow of an ideal incompressible fluid about an aerodynamic cascade with oscillating blades. We shall introduce the following supplementary restrictions. The blades are thin, of small curvature and at a small angle of attack so that they can be replaced by flat plates for the formulation of boundary conditions. The amplitudes of blade oscillations are small.

The equations of motion may be linearized. The solution of the problem of the oscillation of sections of small curvature in a cascade can be obtained by adding the solution for steady flow about the given cascade to the solution for the unsteady flow about a cascade of flat plates. The first problem has a complete solution [1]. Only the unsteady flow will be discussed further.

Far ahead of the cascade the flow is considered to be undisturbed and to have a constant prescribed velocity.

The boundary conditions on the sections are that for the fluid particles on them the normal components of velocity and acceleration (the latter in the linearized formulation only) are known at each moment of time.

In addition, we shall assume that the Chaplygin-Zhukovskii condition must be satisfied at the trailing edges of the oscillating sections. Wakes are shed from the trailing edges of the sections in the case of unsteady motion according to the theorem of constancy of circulation. In passing across the wake there is a jump in velocity, although the pressure field is continuous. The formulation of the Chaplygin-Zhukovskii condition in this case is equivalent to requiring continuity of the pressure function p at the trailing edges of the sections.

The problem consists in determining the velocity and pressure fields as well as the unsteady forces and moments acting on the oscillating sections.

We shall introduce the complex acceleration potential

$$w = \varphi + i\psi = f(z) \tag{1.1}$$

Here ϕ is the acceleration potential and $z = x + iy$ is the complex variable in whose plane lies the cascade.

It is well known that the acceleration potential is related to the variable component of the pressure p by the relation

$$\rho\varphi = -p \tag{1.2}$$

Here ρ is the density of the fluid. For the complex acceleration we have

$$a = \frac{dw}{dz} = a_x - ia_y$$

It is obvious that moving upstream from the cascade $p \to 0$ and, consequently, $\phi \to 0$. On the sections of the cascade the normal acceleration component a_y (let the sections be parallel to the x-axis) will be of known magnitude.

Other properties of the complex acceleration potential depend on the conditions of the problem. If, for example, an aerodynamic cascade whose sections are performing synchronous in-phase oscillations is being considered, the acceleration potential must then be a periodic function with period equal to the pitch of the cascade.

2. Some functions used in the solution of the problem. We shall consider the function of the complex variable z and the real constant q

$$F(z, q) = \frac{1}{q} \ln \frac{\cosh qz + \sqrt{\sinh^2 qz - \sinh^2 q}}{\cosh q} \tag{2.1}$$

The function $F(z, q)$ has the following properties used in the solution of the problem:

1) the function $F(z, q)$ is periodic with period $i\pi/q$;

2) the function $F(z, q) \to z - q^{-1} \ln \cosh q + \ldots$ as $z \to \infty$;

3) the function $F(z, q) \to \sqrt{(z^2 - 1)}$ as $q \to 0$;

4) on the segment $y = 0$, $-1 < x < +1$ the function $F(z, q)$ takes the purely imaginary values

$$F(z, q) = i \frac{1}{q} \sin^{-1} \frac{\sqrt{\sinh^2 q - \sinh^2 qx}}{\cosh q} \tag{2.2}$$

On this segment the function $F(z, q)$ can be expanded into a series of the form

$$F(x, q) = i \sum_{n=1}^{\infty} a_n (1 - x^2)^{n/2} \tag{2.3}$$

The coefficients in this expansion will be functions of the parameter q and can be determined with the help of Fourier series theory.

If attention is restricted to only the first term in the expansion, it is then possible to use the following approximation:

$$F(x, q) = i \frac{1}{q} \tan^{-1} \sinh q \sqrt{1 - x^2} \tag{2.4}$$

This approximation will be sufficiently good even for large values of q. The derivative of the function $F(z, q)$ is equal to

$$F'(z, q) = \frac{\sinh qz}{\sqrt{\sinh^2 qz - \sinh^2 q}} \tag{2.5}$$

We shall introduce into the discussion the function

$$\Phi(z, q) = \frac{1}{q} \ln \frac{\sinh qz + \sqrt{\cosh^2 qz + \sinh^2 q}}{\cosh q} \tag{2.6}$$

This function has the following properties:

1) the function $\Phi(z, q)$ is periodic with period $i\pi/q$;

2) the function $\Phi(z, q) \to z - q^{-1} \ln \cosh q + \ldots$ as $z \to \infty$;

3) the relation $\Phi(z + i\pi/2q, q) = F(z, q)$ is valid;

4) the function $\Phi(z, q) \to z$ as $q \to 0$;

5) on the segment $y = 0$, $-1 < x < +1$ the function $\Phi(z, q)$ takes real values.

On this segment the function $\Phi(x, q)$ can be expanded into a series of the form

$$\Phi(x, q) = \sum_{n=1}^{\infty} b_n x^n \qquad (n = 1, 5 \ldots) \tag{2.7}$$

The coefficients b_n depend on the parameter q. The leading terms of the series are equal to

$$\Phi(x, q) = \frac{1}{\cosh q} x + \frac{6q^4}{5! \cosh q} x^5 + \ldots \tag{2.8}$$

As q decreases the second term of the series decreases very quickly, a property which will be exploited in the approximate solutions.

The derivative of the function $\Phi(z, q)$ is equal to

$$\Phi'(z, q) = \frac{\cosh qz}{\sqrt{\cosh^2 qz + \sinh^2 q}} \tag{2.9}$$

We shall further consider the function

$$P(z, q) = \ln \frac{\sinh qz + \sqrt{\sinh^2 qz - \sinh^2 q}}{\sinh q} \tag{2.10}$$

which has the following properties:

1) the function $P(z, q)$ is periodic with period $i\pi/q$;

2) the function $P(z, q) \to \ln(z + \sqrt{(z^2 - 1)})$ as $q \to 0$;

3) on the segment $y = 0$, $-1 < x < +1$ the function $P(x, q)$ takes purely imaginary values.

The derivative of the function $P(z, q)$ is equal to

$$P'(z, q) = \frac{q \cosh qz}{\sqrt{\sinh^2 qz - \sinh^2 q}} \tag{2.11}$$

We will introduce one final function

$$Q(z, q) = \ln i \frac{\cosh qz + \sqrt{\cosh^2 qz + \sinh^2 q}}{\sinh q}, \quad P(z + i\pi/2q, q) = Q(z, q) \tag{2.12}$$

The derivative of the function $Q(z, q)$ is equal to

$$Q'(z, q) = \frac{q \sinh qz}{\sqrt{\cosh^2 qz + \sinh^2 q}} \tag{2.13}$$

Thus, the periodic functions $F(z, q)$ and $P(z, q)$ have been introduced, as well as the functions $\Phi(z, q)$ and $Q(z, q)$ which are obtained from the former by a half-period shift. These functions correspond to the complex velocity potential for the transverse and circulational flow about a cascade of flat plates [1].

The behavior of the functions $F(z, q) - z$ and $\Phi(z, q) - z$ for large values of the modulus of x corresponds to the behavior of the complex potential of a cascade of doublets lying along the y-axis with pitch $i\pi/q$.

The behavior of the derivatives of the functions $P'(z, q)$ and $Q'(z, q)$ at large values of the modulus of x corresponds to the behavior of the complex potential of a cascade of vortices lying along the y-axis with the same pitch $i\pi/q$.

3. The complex acceleration potential for a cascade. Let an ideal incompressible fluid flow about a straight cascade of oscillating flat plates. We shall place the origin of the coordinate system of the plane $z = x + iy$ at the center of one of the flat plates, and we shall direct the axis of the cascade along the y-axis. We shall take the chord of the flat plates equal to $b = 2$ and the pitch of the cascade equal to t.

We shall use the functions introduced above to represent the complex acceleration potential.

If the sections are oscillating synchronously and in phase, the complex acceleration potential can then be expressed by the following series:

$$w = \sum_{n=1}^{\infty} A_n [F(z, q) - z + q^{-1} \ln \cosh q]^n + \\ + iB \left[\cosh q\, F'(z, q) - q^{-1} \sinh q\, P'(z, q) \right] \tag{3.1}$$

or, after simplifying the last term, we obtain

$$w = \sum_{n=1}^{\infty} A_n \left[F(z, q) - z + q^{-1} \ln \cosh q \right]^n + iB \sqrt{\frac{\sinh q (z-1)}{\sinh q (z+1)}} \tag{3.2}$$

The usefulness of such a representation is determined from the following considerations.

1) For $q = \pi/t$ the complex potential has a period equal to the pitch of the cascade it.

2) For the condition that $\operatorname{Im} B = 0$, $\phi = \operatorname{Re} w \to 0$ at infinity upstream of the cascade, i.e. the flow there will be undisturbed ($p \to 0$).

3) At the trailing edges of the sections the Chaplygin-Zhukovskii condition is satisfied, since the function ϕ is not discontinuous at $z = +1 \pm int$.

The constant (with respect to z) coefficients A_n and B must be determined from the boundary conditions according to the prescribed law of variation of velocity and acceleration of the section. The complex acceleration is found by differentiating the series (3.1).

We shall pass on to the consideration of synchronous out-of-phase oscillations. For this type of oscillation the acceleration potential must have a period equal to double the pitch of the cascade. The magnitudes of the normal accelerations and velocities on adjacent blades will be equal in modulus and opposite in sign.

We shall seek a solution for the complex acceleration potential in the form of the series

$$w = \sum_{n=1}^{\infty} A_n \left[\left(F - z + 2q^{-1} \ln \cosh \frac{q}{2} \right)^n - \left(\Phi - z + 2q^{-1} \ln \cosh \frac{q}{2} \right)^n \right] + \\ + iB \left[\cosh \frac{q}{2} (F' - \Phi') - 2q^{-1} \sinh \frac{q}{2} (P' - Q') \right] \tag{3.3}$$

Here, for brevity, the writing of arguments has been omitted, i.e.

$$F = F(z, q/2), \qquad \Phi = \Phi(z, q/2), \qquad P = P(z, q/2), \qquad Q = Q(z, q/2)$$

The usefulness in expressing w by a series of form (3.3) is indicated by the following considerations.

The period of the functions F, Φ, P and Q which occur in series (3.3) is equal to $2i\pi/q$. If we take $q = \pi/t$, the period of the functions is then equal to $2it$, i.e. to double the pitch of the cascade. As y varies over a half-period, i.e. over the magnitude of the pitch of the cascade it, the functions u and a change sign.

As the pitch of the cascade increases without limit $t \to \infty (q \to 0)$ the expressions for the complex acceleration potential (3.1) and (3.3) reduce to the series

$$w = \sum_{n=1}^{\infty} A_n (\sqrt{z^2 - 1} - z)^n + iB \sqrt{\frac{z-1}{z+1}} \tag{3.4}$$

4. Determination of the added masses of the oscillating sections in an aerodynamic cascade. For the case in which the sections in an aerodynamic cascade are oscillating but the cascade is not immersed in a flow, only the pressure which corresponds to the inertial forces of the fluid surrounding the cascade acts on the sections The action of these forces can be accounted for by means of the added masses.

For the case in which flat plates are performing synchronous in-phase oscillations and the flow velocity far upstream of the cascade is $U = 0$, the complex acceleration potential can be expressed by series (3.1), after setting $B = 0$ in it:

$$w = i \sum_{n=1}^{\infty} A_n \left[F(z, q) - z + q^{-1} \ln \cosh q \right]^n \tag{4.1}$$

In order that the period of this function be equal to the period of the cascade, it is necessary to set $q = \pi/t$. The imaginary unit i is supplied in front of the entire series and all the coefficients A_n will now be real numbers (relative to i) in view of the symmetry of the problem.

The complex acceleration is expressed by the following series:

$$a = i\,[F'(z, q) - 1] \sum_{n=1}^{\infty} nA_n \left[F(z, q) - z + q^{-1} \ln \cosh q \right]^{n-1} \tag{4.2}$$

The coefficients A_n of the series can be found since the law of oscillation of the flat plates is known, i.e. the function $a_y = a_y(x, \tau)$, where τ is the time is known.

We shall consider the determination of the added masses for some special cases.

a) Synchronous in-phase bending oscillations of blades. We shall refer to those oscillations in which the sections move in a direction perpendicular to the chord as bending oscillations.

Let the flat plates oscillate according to the harmonic law

$$v = v_0 \exp j\omega\tau$$

where v is the velocity of the oscillations, ω is the frequency of the oscillations, and j is an imaginary unit which does not interact with the imaginary unit i.

The acceleration in the direction of the y-axis is found by differentiation:

$$a_y = j\omega v_0 \exp j\omega\tau$$

For the special case under consideration, series (4.1) is limited to only the first term.

This follows from the fact that the complex acceleration

$$a = a_x - ia_y = iA\ [F'(z, q) - 1]$$

with $A = j\omega v_0 \exp j\omega t$ meets the boundary conditions on the flat plate contour since $F'(x, q)$ takes imaginary values on the flat plate.

We shall further find the pressure distribution on the flat plate

$$p = A\ \rho q^{-1} \sin^{-1} \frac{\sqrt{\sinh^2 q - \sinh^2 qx}}{\cosh q}$$

The force acting on the flat plate is found by integrating p over the flat-plate contour. Since the force is found to be in phase with the acceleration it can be replaced by the effect of an added mass. In the case under consideration an exact solution can be obtained which is the well-known one [1]

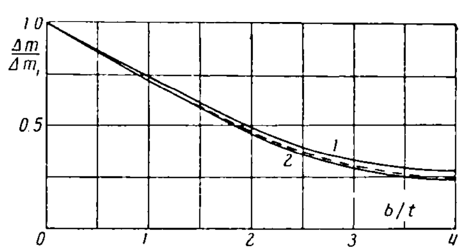

Fig. 1.

$$\Delta m = \rho \frac{2t^2}{\pi} \ln \cosh \frac{\pi b}{2t} \tag{4.3}$$

We shall also obtain an approximate solution, depending on an approximate representation of the function $F(x, q)$ on the segment $-1 \leqslant x \leqslant +1$ by only one term of the series (2.4). Omitting the elementary integration, we shall present the approximate formula for added mass

$$\Delta m = \rho \frac{bt}{2} \tan^{-1} \sinh \frac{\pi b}{2t} \tag{4.4}$$

As $b/t \to 0$ the approximate solution tends to

$$\Delta m_1 = \rho\pi \frac{b^2}{4} \tag{4.5}$$

i.e. it coincides with the exact value of the added mass for an isolated flat plate.

A comparison of the exact solution (4.3) (curve 1) and the approximate solution (4.4) (curve 2) is given in Fig. 1.

In the final formulas it has been convenient to abandon the condition $b = 2$ and to take, as has been done both here and later, an arbitrary chord b.

b) Synchronous in-phase torsional oscillations of blades. We shall consider the synchronous in-phase harmonic torsional oscillations of flat plates about their centers

$$v = v_0 x \exp j\omega\tau$$

The acceleration component along the y-axis must be equal to

$$a_y = j\omega v_0 x \exp j\omega\tau \tag{4.6}$$

In this case it is necessary to use the series (4.1) to represent the acceleration potential and the complex acceleration. In view of the fact that the acceleration a_y must be an odd function of x, only those terms with coefficients A_n having even indices should be retained in the series.

We shall obtain an approximate solution after using an approximate representation of the function $F(x, q)$ on the segment $-1 \leqslant x \leqslant +1$ with the help of Expression (2.4).

In this case the series is limited to only the first term with an even index, and it is easy to obtain

$$A_2 = -\frac{1}{2} v_0 j\omega \exp j\omega\tau \frac{q^2}{q^2 + (\tan^{-1} \sinh q)^2} \tag{4.7}$$

The value of the acceleration potential of the flat plate is then found in the usual way, and after integrating according to the formula

$$\Delta m = -2 \int_{-b/2}^{+b/2} \rho \varphi x dx \tag{4.8}$$

the value of the generalized added mass of the sections in a cascade is found to be

$$\Delta m = \frac{\pi \rho b^4}{64} \frac{q \tan^{-1} \sinh q}{q^2 + (\tan^{-1} \sinh q)^2}, \qquad q = \frac{\pi b}{2t} \tag{4.9}$$

As $b/t \to 0$ this expression becomes

$$\Delta m_1 = \frac{\pi \rho b^4}{128} \tag{4.10}$$

which coincides with the well-known exact solution for an isolated flat plate which is performing rotary motions about its center.

For the rotation of flat plates not about their centers it is possible to use (4.3), (4.9) and the principle of superposition.

We shall pass on to the consideration of synchronous out-of-phase oscillations of sections. For this type of oscillation the acceleration potential must have a period equal to double the pitch of the cascade. The values of the normal accelerations on adjacent blades must be equal in modulus and opposite in sign.

We shall seek a solution for the complex acceleration potential in the form of the series (3.3), setting $B = 0$ in it (for $U = 0$ the boundary conditions on the section are satisfied by choosing the coefficients A_n).

In view of the symmetry of the problem we shall provide the imaginary unit i in front of the entire series; all the A_n will be real (relative to i) numbers.

We then obtain

$$w = i \sum_{n=1}^{\infty} A_n \left\{ \left[F(z, \tfrac{q}{2}) - z + \frac{2}{q} \ln \cosh \frac{q}{2} \right]^n - \right.$$
$$\left. - \left[\Phi(z, \tfrac{q}{2}) - z + \frac{2}{q} \ln \cosh \frac{q}{2} \right]^n \right\} \tag{4.11}$$

We shall continue the special cases.

c) Synchronous out-of-phase bending oscillations. Let adjacent flat plates oscillate out of phase:

$$v = v_0 \exp j\omega\tau, \qquad v = -v_0 \exp j\omega\tau \tag{4.12}$$

Confining ourselves to approximate representation of the functions, we shall obtain an approximate solution of the problem for small values of q. In this case the coefficient A_1 in the series (4.11) is defined by the following expression:

$$A_1 = j\omega v_0 \exp j\omega\tau \cosh (q/2)$$

The determination of the acceleration potential and the calculation of the added mass is carried out according to the usual formulas. We shall present the final expression for the added mass

$$\Delta m = \rho bt \cosh \frac{\pi b}{4t} \tan^{-1} \sinh \frac{\pi b}{4t} \tag{4.13}$$

As $b/t \to 0$ we obtain the well-known expression (4.5) for the added mass of an isolated flat plate.

It is obvious that for out-of-phase oscillations the added mass of a flat plate in a cascade is larger than the added mass of an isolated flat plate.

d) Synchronous out-of-phase torsional oscillations. Let adjacent flat plates rotate out of phase about their centers according to the harmonic law

$$v = v_0 x \exp j\omega\tau, \qquad v = -v_0 x \exp j\omega\tau \tag{4.14}$$

Using the same dependencies as in the previous paragraph, it is possible to obtain the following approximate expression for the generalized added mass:

$$\Delta m = \frac{\pi\rho b^4}{64} \frac{(q/2)\cosh^2 q/2 \tan^{-1} \sinh q/2}{(q/2)^2 + (\tan^{-1} \sinh q/2\cosh q/2)^2}, \quad q = \frac{\pi b}{2t} \tag{4.15}$$

Here only one term of the series (4.11) has been used; a more accurate solution is easily obtained by taking additional terms into account.

5. Flow about a cascade of oscillating sections. We shall consider the flow of an ideal incompressible fluid about a straight cascade (without stagger) of oscillating flat plates. At infinity upstream of the cascade the flow has the velocity U.

As before, the coefficients A_n are determined according to the prescribed acceleration a_y of the flat plate. The coefficient B, which is now not equal to zero, does not enter into these calculations, since the expression by which it is multiplied will give only the acceleration component a_x. This is obvious from consideration of the expression which represents the derivative of the last term in (3.2)

$$\frac{iqB\sinh 2q}{2\sinh q\,(z+1)\sqrt{\sinh q\,(z-1)\sinh q\,(z+1)}} \tag{5.1}$$

This expression takes real values (relative to i) on the segment $y = int$, $-1 \leqslant x \leqslant +1$.

The coefficient B must be chosen to satisfy the kinematic condition on the oscillating contour, where the normal velocity component of the fluid particles is determined by the impenetrability condition of the contour.

The expression (5.1) has an imaginary (relative to i) value on the semi-infinite straight lines $y = int$, $x < -1$ and, consequently, it affects the acceleration component a_y and by the same token the velocity v of the fluid particles which run from infinity to the oscillating sections of the cascade.

The acceleration component of the fluid particles along the y-axis can be represented by the sum of the local and convective accelerations.

Linearizing the equations of motion, we obtain

$$a'_y = \frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} \tag{5.2}$$

We shall consider harmonic oscillations and shall express the velocity and acceleration functions in time

$$a'_y = a_y \exp j\omega\tau, \qquad v' = v \exp j\omega\tau$$

Here a_y and v will be functions of x and y. The function a_y can have complex values (relative to j).

In place of (5.2) we finally obtain

$$a_y = j\omega v + U\partial v/\partial x \tag{5.3}$$

This equation can be written more conveniently in the complex (relative to i) form

$$a = j\omega c + U\partial c/\partial x \tag{5.4}$$

where a and c are the complex acceleration and velocity, respectively.

Integrating this linear equation under the conditions that $v = 0$ far upstream of the cascade and that $v = iv_0 \exp j\omega t$ on the sections in the cascade, we obtain

$$iv_0 \exp j\omega\tau = \frac{1}{U}e^{jk}\int_{-\infty}^{-1} ae^{jkz}dz, \qquad k = \frac{\omega b}{2U} \tag{5.5}$$

The upper limit of the integral is taken equal to -1, since the chord length of the section is $b = 2$.

The quantity B, which has been discussed above, must be determined with the help of condition (5.5).

The parameter k is the Strouhal number.

The parameter k can be treated as the ratio of the mean vorticity in the wake $\omega\Gamma/\pi U$ generated per half-cycle of oscillation to the mean vorticity at the section Γ/b (Γ is the velocity circulation about the section created by the oscillation).

a) Synchronous in-phase bending oscillations of blades. For this special case the exact expression for the complex potential can be written from (3.1):

$$w = iA\ [F\ (z,\ q) - z] + iB\ [\cosh qF'\ (z,\ q) - q^{-1} \sinh qP'\ (z,\ q)] \quad (5.6)$$

$$A = jkUv_0 \exp j\omega\tau$$

Here the quantity A is a real constant (relative to i and z).

Differentiating this expression, we find the complex acceleration

$$a = iA\ [F'\ (z,\ q) - 1] + iB\ [\cosh qF''\ (z,\ q) - q^{-1} \sinh qP''\ (z,\ q)] \quad (5.7)$$

Substituting a according to (5.6) into (5.5) and carrying out the integration of the second term by parts, we obtain

$$B = -R\ (k,\ q)\ v_0 U \exp j\omega\tau \quad (5.8)$$

Here $R(k,\ q)$ is a function complex in j and equal to

$$R\,(k, q) = \frac{1 + jke^{jk} J_1\,(k,\ q)}{1 + jke^{jk} J_2\,(k,\ q)} \quad (5.9)$$

Moreover, J_1 and J_2 are the improper integrals

$$J_1\,(k,\ q) = \int_1^\infty \left(\frac{\sinh qx}{\sqrt{\sinh^2 qx - \sinh^2 q}} - 1 \right) e^{-jkx} dx \quad (5.10)$$

$$J_2\,(k,\ q) = \int_1^\infty \left(\sqrt{\frac{\sinh q\,(x+1)}{\sinh q\,(x-1)}} - e^{q} \right) e^{-jkx} dx \quad (5.11)$$

As the pitch of the cascade increases without limit ($t \to \infty$) $q \to 0$ and the integrals (5.10) and (5.11) simplify respectively to the integrals

$$J_1(k, 0) = \int_1^\infty \left(\frac{x}{\sqrt{x^2-1}} - 1 \right) e^{-jkx}\, dx$$
$$J_2(k, 0) = \int_1^\infty \left(\sqrt{\frac{x+1}{x-1}} - 1 \right) e^{-jkx}\, dx \tag{5.12}$$

which can be expressed by the Hankel functions [16]

$$J_1(k, 0) = -\frac{\pi}{2} H_1^{(2)}(k) - \frac{1}{jk} e^{-jk}$$
$$J_2(k, 0) = -\frac{\pi}{2} [H_1^{(2)}(k) + jH_0^{(2)}(k)] - \frac{1}{jk} e^{-jk} \tag{5.13}$$

Substituting (5.13) into (5.9) we are convinced that the function $R(k, q)$ in the limiting case of $q = 0$ transforms into the well-known Theodorsen function $C(k)$ which plays an important role in investigating the oscillations of an isolated section

$$R(k, 0) = C(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + jH_0^{(2)}(k)} \tag{5.14}$$

We shall pass on to the calculation of the forces which act on an oscillating section in a cascade.

The pressure distribution on the section is found by extracting the real (with respect to i) part of (5.6). After substituting the values of the functions $F(x, q)$ and $P(x, q)$, we obtain

$$p = \frac{1}{q} A\rho \sin^{-1} \frac{\sqrt{\sinh^2 q - \sinh^2 qx}}{\cosh q} + \rho B \sqrt{\frac{\sinh q(1-x)}{\sinh q(1+x)}} \tag{5.15}$$

The pressures on the upper and lower sides of the section are equal in magnitude but opposite in sign.

The force acting on the blade is found by integrating (5.15) over the blade contour.

The force obtained from integrating the first term in (5.15) is not related to the circulation but is equal to the added mass multiplied by the acceleration. The calculation of the added mass for the type of oscillation under consideration has already been carried out ((4.3) or (4.4)).

Integrating the second term in (5.15) gives the force component which does depend on the circulation

$$L = -4\rho v_0 U \frac{t}{b} \sinh \frac{\pi b}{2t} R(k, q)\, e^{j\omega\tau} \tag{5.16}$$

In the final formula we introduce here, as also earlier, the section chord b. As the pitch of the cascade increases without limit ($t/b \to \infty$) Formula (5.1) transforms into the well-known formula for the lift force of an isolated oscillating flat plate

$$L = -2\pi\rho v_0 U C(k)\, e^{j\omega\tau}$$

b) Bending oscillations of blades in a cascade with different frequencies, phases and amplitudes. We shall consider the problem of flow about a cascade whose adjacent sections are oscillating with different frequencies and phases, but such that all the odd sections have the frequency ω_1 and the even ones have the frequency ω_2 and the phase shift θ. We shall designate the oscillation velocity of the sections by v_1 for the odd sections and by v_2 for the even ones.

Thus, the odd sections, including the one which lies at the origin of the coordinate system, oscillate according to the law

$$v_1 = v_{01} \exp j\omega_1\tau \tag{5.17}$$

The even sections oscillate also, according to the law

$$v_2 = v_{02} \exp j(\omega_2\tau - \theta) \tag{5.18}$$

The expression for the complex acceleration potential can be constructed according to the type of (3.3), but the terms with the functions F and Φ must be grouped in different series, because adjacent sections have different characteristics of the oscillating process.

We shall consider the problem in approximate formulation and limit ourselves to only the first terms in the series. We seek the complex potential in the form

$$\begin{aligned} w = {} & iA\,[F(z, q/2) - z] + iB\,[\Phi(z, q/2) - z] + \\ & + iC\,[\cosh q/2F'(z, q/2) - 2q^{-1}\sinh q/2P'(z, q/2)] + \\ & + iD\,[\cosh q/2\Phi'(z, q/2) - 2q^{-1}\sinh q/2Q'(z, q/2)] \end{aligned} \tag{5.19}$$

This expression can be written as

$$\begin{aligned} w = {} & iA\,[F(z, q/2) - z] + iB\,[\Phi(z, q/2) - z] + \\ & + iC\sqrt{\frac{\sinh {}^1/_2\, q\,(z-1)}{\sinh {}^1/_2\, q\,(z+1)}} + iD\,\frac{\cosh {}^1/_2\, q\,(z-1)}{\sqrt{\cosh^2 {}^1/_2\, qz + \sinh^2 {}^1/_2\, q}} \end{aligned} \tag{5.20}$$

Differentiating (5.20) with respect to z, we obtain the complex acceleration a. Differentiating (5.17) and (5.18) with respect to τ and taking into account that $\partial v/\partial x = 0$ on flat plates in the problem under consideration, we obtain the values of the normal accelerations $a_1 = j\omega_1 v_{01} \exp j\omega_1 \tau$ and $a_2 = j\omega_2 v_{02} \exp j(\omega_2\tau - \theta)$. Equating corresponding values of the accelerations, we arrive at two equations which relate the desired quantities A, B, C and D:

$$\begin{aligned} -a_1 &= -A + [\operatorname{sech}(q/2) - 1]\, B + [1 - q/2 \tanh(q/2)]\, D \\ -a_2 &= [\operatorname{sech}(q/2) - 1]\, A - B + [1 - q/2 \tanh(q/2)]\, C \end{aligned} \tag{5.21}$$

In the derivation an approximate representation of the functions $\Phi(z, q/2)$ and $Q'(z, q/2)$ on the segment $-1 \leqslant x \leqslant +1$ by the first terms of the series has been used.

Two other equations relating the coefficients must be determined such that the value of the normal velocities on the flat plates and the condition that the flow velocity at infinity upstream of the cascade is equal to $U(v = 0)$ are satisfied.

Using the boundary conditions for the velocity on the odd sections, with the help of the integral (5.5) we find

$$v_1 U = -e^{jk_1} AJ_1 - e^{jk_2} BJ_3 + C\,(1 + e^{jk_1} jk_1 J_2) + D\,(1 + e^{jk_1} jk_2 J_4) \tag{5.22}$$

Here $J_1 = J_1(k, q/2)$ and $J_2 = J_2(k, q/2)$ are the integrals (5.10) and (5.11) which have been introduced above, and $J_3 = J_3(k, q/2)$ and $J_4 = J_4(k, q/2)$ are also functions of k and q which are expressed by the integrals

$$J_3(k, q) = \int_1^\infty \left(\frac{\cosh qx}{\sqrt{\cosh^2 qx + \sinh^2 q}} - 1 \right) e^{-jkx}\, dx \tag{5.23}$$

$$J_4(k, q) = \int_1^\infty \left(\frac{\cosh (x + 1)}{\sqrt{\cosh^2 qx + \sinh^2 q}} - eq \right) e^{-jkx}\, dx \tag{5.24}$$

To satisfy the boundary conditions for the velocity on the even sections an integral of the type (5.5) along the semi-infinite straight line $y = it$, $-\infty \leqslant x \leqslant -1$ must be calculated. Using the periodicity property of the functions F, Φ, P and Q, we obtain the following condition:

$$v_2 U = -e^{jk_1} AJ_3 - e^{jk_2} BJ_1 + C\,(1 + e^{jk_1} jk_1 J_4) + D\,(1 + e^{jk_1} jk_2 J_2) \tag{5.25}$$

Solving Equations (5.21), (5.22) and (5.25), we find A, B, C and D.

The pressure-distribution function on the oscillating sections is found by extracting the real (with respect to i) part of the complex acceleration potential (5.20). We obtain Expression (5.15), only $q/2$ must be substituted in place of q and C in place of B.

In the general case, forces having the frequencies ω_1 and ω_2 act on the section. The action of the forces which coincide in phase and frequency with the accelerations can be replaced by the effect of an added mass.

In addition, forces which coincide in frequency and phase with the accelerations of adjacent sections but are not related to the variation of the circulation of the section under consideration also act on the section.

In the case for which the oscillation frequencies of adjacent flat plates are equal and the phase shift is equal to 0 or π, the action of these forces can also be reduced to the effect of an added mass. This is also possible in the case for which the even (or odd) sections are motionless ($\omega_1 = 0$ or $\omega_2 = 0$).

The added masses are determined by a formula of type (4.4). We shall give as an example the formula for the added mass of a flat plate in a cascade in which the oscillating and motionless flat plates ($U = 0$) are alternated:

$$\Delta m = \frac{1}{4} \frac{\rho b t}{1-(\operatorname{sech} q/2 - 1)^2} \tan^{-1} \sinh q/2 \tag{5.26}$$

Integrating the second term of (5.15) and substituting the coefficient C in place of B, and $q/2$ in place of q, as had been said, we obtain the force component associated with the variation of the circulation

$$L = 8\rho C \frac{t}{b} \sinh \frac{\pi b}{4t} \tag{5.27}$$

This force acts on the odd sections. For calculating the force acting on the even sections, C must be replaced by D.

c) Synchronous in-phase torsional oscillations of blades in a cascade. We shall consider the flow about a straight cascade whose blades are performing torsional oscillations according to the law

$$y = y_0 x \exp j\omega\tau$$

The normal component of the flow velocity on the section depends on the instantaneous velocity and the instantaneous position of the flat plate

$$v = y_0 U e^{j\omega\tau} (1 + jkx) \tag{5.28}$$

The normal component of the flow acceleration on the section is found with the help of (5.3)

$$a_y = j y_0 U^2 k e^{j\omega\tau} (2 + jkx) \tag{5.29}$$

We shall show that even for the solution of the problem of thick cascades one can confine oneself to the first two terms with A_n in the complex acceleration potential

$$w = i \sum_{n=1}^{2} A_n \left[F(z, q) - z + q^{-1} \ln \cosh q\right]^n + iB \sqrt{\frac{\sinh q (z-1)}{\sinh q (z+1)}} \quad (5.30)$$

where A_n and B are real numbers.

The complex acceleration is found by differentiating (5.30) with respect to z

$$a = \frac{dw}{dz} = iA_1 \frac{dw_1}{dz} + iA_2 \frac{dw_2}{dz} + iB \frac{dw_3}{dz} \quad (5.31)$$

where w_1, w_2, w_3 are the corresponding parts of the complex acceleration potential (5.30).

The normal component of the accelerations on the sections is found by extracting the imaginary part of Expression (5.31). The third term in (5.31) has been found above in (5.1) and will be real for $y = 0$, $-1 \leqslant x < +1$.

The imaginary part of the first and second terms in (5.31) is easily found, using Expressions (2.1), (2.2) and (2.5):

$$a_y = -A_1 - 2A_2 \left(\frac{\sinh qx}{q\sqrt{\sinh^2 q - \sinh^2 qx}} \sin^{-1} \frac{\sqrt{\sinh^2 q - \sinh^2 qx}}{\cosh q} + x \right) \quad (5.32)$$

In the limiting case of a cascade of infinitely large pitch ($q = 0$) the expression in parentheses simplifies to $2x$, i.e. it gives the exact solution for the problem of an isolated wing.

It is however easily observed that the function in parentheses is very nearly linear, i.e. it practically gives the exact solution even in the case of a thick cascade. This is confirmed by the graph presented in Fig. 2, where the value of a_{y2} is plotted as a fraction of its value at the edges of the flat plates

$$a_{y2}(1) = 2\left(1 + \frac{1}{q} \tanh q\right)$$

This was shown earlier by applying the approximate dependency (2.4).

The coefficients A_n in the expression for the complex potential are then determined according to the known acceleration

$$A_1 = -2jy_0 kU^2 e^{j\omega\tau}$$

$$A_2 = \frac{1}{2} \frac{q}{q + \tanh q} y_0 k^2 U^2 e^{j\omega\tau} \quad (5.33)$$

The coefficient B in (5.30) must be determined so that the boundary conditions for the normal velocity on the flat plates are satisfied.

Applying Expression (5.5) for the case under consideration and using (5.30), (5.31), (2.1) and (5.33), we obtain the value of

$$B = -y_0 U^2 e^{j\omega\tau} \frac{1 - jk + 2jke^{jk} J_1(k, q) + \frac{1}{2} jk^3 q (q + \tanh q)^{-1} J_5(k, q)}{1 + jke^{jk} J_2(k, q)} \tag{5.34}$$

Here the functions $J_1(k, q)$ and $J_2(k, q)$ are expressed by the integrals (5.10) and (5.11), and the function $J_5(k, q)$ by the integral

$$J_5(k, q) = \int_1^{\infty} \left[\frac{1}{q} \ln\left(\cosh qx + \sqrt{\sinh^2 qx - \sinh^2 q}\right) - x\right]^2 e^{-jkx} dx \tag{5.35}$$

The pressure distribution on the oscillating sections is found by extracting the real (with respect to i) part of (5.30)

$$p = \rho(A_1 - 2A_2 x) \frac{1}{q} \sin^{-1} \frac{\sqrt{\sinh^2 q - \sinh^2 qx}}{\cosh q} + \rho B \sqrt{\frac{\sinh q(1-x)}{\sinh q(1+x)}} \tag{5.36}$$

The integration of the first term of (5.36) on the contour of the flat plate gives a force whose action reduces to the effect of a generalized added mass.

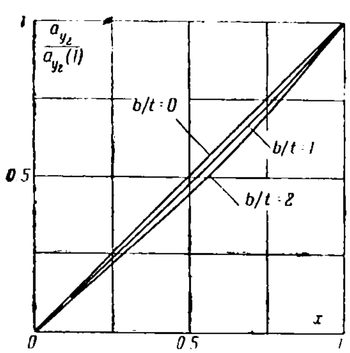

Fig. 2.

The integration of the second term gives the force component which depends on the circulation

$$L = 4\rho B \frac{t}{h} \sinh \frac{\pi b}{2t} \tag{5.37}$$

where B is given by Formula (5.34). The integration of (5.36) in accordance with Formula (4.8) determines the moment which is acting.

d) Torsional oscillations of blades in a cascade with different frequencies, phases and amplitudes. We shall consider only that special case for which all odd sections oscillate according to the law

$$y = y_{01} x \exp j\omega_1 \tau \tag{5.38}$$

and all even ones according to the law

$$y = y_{02} x \exp j(\omega_2 \tau - \theta) \tag{5.39}$$

In the solution we shall limit ourselves to only the first terms of the series for the complex acceleration potential

$$\acute{w} = i \sum_{n=1}^{2} A_n \left[F\,(z,\, q/2) - z + \frac{2}{q} \ln \cosh \frac{q}{2} \right]^n + i \sum_{n=1}^{2} B_n \left[\Phi\left(z,\, \frac{q}{2}\right) - z + \right.$$

$$\left. + \frac{2}{q} \ln \cosh \frac{q}{2} \right]^n + iC \sqrt{\frac{\sinh {}^1\!/_2\, q\,(z-1)}{\sinh {}^1\!/_2\, q\,(z+1)}} + iD \frac{\cosh {}^1\!/_2\, q\,(z-1)}{\sqrt{\cosh^2 {}^1\!/_2\, qz + \sinh^2 {}^1\!/_2\, q}} \qquad (5.40)$$

The normal components of the acceleration of the flow on the flat plates must be equal to

$$\begin{aligned} a_{y1} &= a_1 + a_2 x = j y_{01} U^2 k_1 e^{j\omega_1 \tau}\,(2 + jk_1 x) \\ a_{y2} &= a_3 + a_4 x = j y_{02} U^2 k_2 e^{j(\omega_2 \tau - \theta)}\,(2 + jk_2 x) \end{aligned} \qquad (5.41)$$

Differentiating (5.40) with respect to z, we shall find the expression for the complex acceleration. Equating the normal components of the acceleration, we shall find equations which relate the coefficients A_1, B_1, C and D (taking the first terms of the series of the approximate representation of the functions F, Φ and Q' on the segment $-1 \leqslant x \leqslant +1$):

$$\begin{aligned} -a_1 &= A_1 + \alpha B_1 + \beta D, \\ -a_3 &= \alpha A_1 + B_1 + \beta C, \end{aligned} \qquad \left(\alpha = \operatorname{sech} \frac{q}{2} - 1,\ \beta = 1 - \frac{q}{2} \tanh \frac{q}{2}\right) \quad (5.42)$$

Two more equations relating the coefficients A_1, B_1, C and D are found from the kinematic conditions, just as was done in part (b) of this section.

The coefficients A_2 and B_2 are found directly from the known values of a_2 and a_4

$$A_2 = \frac{1}{2} \frac{-\gamma a_2 + \alpha^2 a_4}{\gamma^2 - \alpha^4}, \qquad B_2 = \frac{1}{2} \frac{-\gamma a_4 + \alpha^2 a_2}{\gamma^2 - \alpha^4} \qquad \left(\gamma = 1 + \left[\frac{2}{q} \tan^{-1} \sinh \frac{q}{2}\right]^2\right) \qquad (5.43)$$

6. Calculation of the flow about a skewed cascade of oscillating sections. The calculation of the flow about a skewed cascade of oscillating sections can be reduced to the calculation of a straight one using the method of conformal transformation.

We shall consider a skewed cascade of flat plates parallel to the abscissa axis of the complex plane $\zeta = \xi + i\eta$. The axis of the cascade is inclined by an angle β to the abscissa axis.

The conformal transformation of the straight cascade in the plane $z = x + iy$ to a skewed one is given by the function

$$\zeta = \sin \beta z - i \cos \beta F\,(z,\, q) \qquad (6.1)$$

The pitch of the skewed cascade is equal to $te^{i\beta}$, where t is the pitch of the straight cascade. The critical points, where the derivative $d\zeta/dz = 0$, correspond to the sharp edges of the skewed cascade in the z-plane. The coordinates of the critical points $x_{1,2}$ are found with the help of (6.1) and (2.1):

$$\sinh qx_{1,2} = \pm \sin\beta \sinh q \tag{6.2}$$

The complex acceleration potential $w = \phi + i\psi$ must be determined in the parametric plane z so that: (1) $\phi \to 0$ as $x \to -\infty$; (2) the value of the imaginary part of the derivative of the complex potential on the sections of the parametric cascade is defined by the condition

$$\operatorname{Im} dw / dz = -a_y d\zeta / dz$$

where a_y is the normal component of the acceleration of the fluid on the sections of the skewed cascade; (3) at the points which correspond to the trailing edges of the sections of the skewed cascade in the z-plane $\phi = 0$.

We shall give as an example the complex acceleration potential for two special cases.

a) Synchronous in-phase bending oscillations of blades. The complex acceleration potential which satisfies the prescribed requirements has the form

$$w = Ae^{-i\beta}\left[F(z, q) - z - q^{-1}\ln\cosh q\right] + iB[\sqrt{1+\sin^2\beta\sinh^2 q}\; F'(z,q) - q^{-1}\sin\beta\sinh q\, P'(z, q)] \tag{6.3}$$

The real (with respect to i and z) constants A and B are determined in the same way as before, only it is necessary to take into account that the complex acceleration in the ζ-plane is now equal to $a(\zeta) = dw/dz : dz/d\zeta$.

b) Synchronous out-of-phase bending oscillations of blades. The complex acceleration potential in the parametric plane is given by the following expression (for the same stipulations as in Section 5):

$$w = A\,[F - \Phi](\cos\beta - i\sin\beta\cosh q/2) + iB[\sqrt{1+\sin^2\beta\sinh^2 q/2}\;(F' - \Phi') + q^{-1}\sin\beta\sinh q\,(P' - Q')] \tag{6.4}$$

For brevity in writing, the arguments have been omitted here, i.e. $F = F(z, q/2)$, $\Phi = \Phi(z, q/2)$, etc.

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